Analysis of Plates under Hydrostatic and Patch Loading using Integrated Force Method

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***Abstract*-- Analysis of transversely loaded rectangular plates is presented by an approach which combines equilibrium equations and compatibility conditions in a single set of equations to provide results for moments and deflections simultaneously. A rectangular element with 9 force and 12 displacement degrees of freedom, which is developed based on Integrated Force Method, is employed to discretize the rectangular plates subjected to patch and hydrostatic loading. Numerical results are presented in tabular and graphical forms and are compared with those based on classical formulation. A good agreement is found when 5 x 5 discretization is used for quarter plate, due to two way symmetry, in case of patch loading and when 5 x 10 mesh is used for half plate, due to one way symmetry, in case of hydrostatic loading.**

***Index Terms*— IFM, Hydrostatic loading, Patch loading, Plate bending, Matlab.**

I. INTRODUCTION

THIN plates are the key structural elements in many engineering applications. Ships, box girders, plate girders aircraft and containers are few of the major large-scale structures using metal plates. A non-metallic plate, such as sheet glass and plywood also has a wide application in lighter structures. The usual function of a thin plate is to withstand a transverse loading, or to act with adjoining structure in sustaining in-plane forces, or both. This paper deals with transverse loading alone with the out of plane deformation being so small that it is good enough to consider only small deflection theory. In other words, it is based on Kirchoff's thin plate theory where it is assumed that the lateral loads are supported through bending action only and membrane forces are absent.

The problem of thin rectangular plates is well known in classical theory of elasticity and a number of analytical methods are well discussed in number of books [1-3]. With the availability of computers, the numerical methods such as finite difference method, finite element method, finite difference energy method and discrete energy method have been widely used for plate analysis [4-6].

A numerical method of analysis, known as Integrated Force Method (Patnaik [7]), which is based on coupling of equilibrium matrix and compatibility matrix in the single matrix, is employed in the present paper to predict the behavior of rectangular plates under patch loading. Such loading is frequently encountered in practice, e.g., wheel load on man hole covers, wheel load on crane girders, diaphragms on bearing of box girders, and during the process of erecting large plate- and box- girder bridges by launching. External loads may be distributed over a wide range of distances varying from over the whole area to over an extremely narrow area. Also, in the present study, plate subjected to hydrostatic loading (linearly varying distribution of transverse pressure), which is quite common in liquid container, is considered for the analysis.

After giving elementary theory of integrated force method and solution steps, results obtained for a number of simply supported rectangular plates under two different types of loading i.e. patch loading and hydrostatic loading are presented in tabular and graphical forms with comparison of results with those based on classical methods [2, 3]

II. ELEMENTARY THEORY OF IFM

The IFM equations for a continuum discretized by finite number of elements with 'n' and 'm' force and displacement degrees of freedom respectively, are obtained by coupling the 'm` number of equilibrium equations and r = n – m compatibility conditions. The m equilibrium equations (EEs) are written as

$$[B]\,\{F\} = \{P\} \tag{1}$$

and the' r' compatibility conditions are written as

$$[C]\,[G]\,\{F\} = \{\delta R\} \tag{2}$$

These conditions are combined to obtain basic fundamental equation as follows:

$$\left[\frac{[B]}{[C][G]}\right]\{F\} = \left[\frac{\{P\}}{\{\delta R\}}\right] \quad \text{Or } [S]\,\{F\} = \{P\} \tag{3}$$

The displacements {X} are back calculated using the following equation,

$$\{X\} = [J]\,\{[G]\,\{F\} + \{\beta^0\}\} \tag{4}$$

where, $[J]$ = m rows of $[[S]^{-1}]^T$, [B] is of m x n size rectangular matrix which is sparse and unsymmetrical, [G] is the symmetrical flexibility matrix; it is a block-diagonal matrix where each block represents a flexibility matrix for an element, [C] is the compatibility matrix of size r x n, $\{\delta R\} = -[C]\,\{\beta\}^0$ is the effective deformation vector with $\{\beta\}^0$ being the initial deformation vector of dimension 'n', here its value equals to zero, [S] is the IFM governing unsymmetrical matrix of size n x n, [J] is the m x n size deformation coefficient matrix which is back-calculated from [S] matrix.

The IFM formulation procedure requires calculation of following three matrices: Equilibrium matrix [B] which links internal forces to external loads, Compatibility matrix [C] which governs the deformations and Flexibility matrix [G] which relates deformations to forces. Both equilibrium and compatibility matrices of the IFM are unsymmetrical, whereas the material constitutive and the flexibility matrices are symmetrical. The complete formulation of above matrices for a rectangular plate bending element is given elsewhere (Doiphode et al. [8]) and hence it is not repeated here.

III. SOLUTION STEPS

Step 1: A four-nodded rectangular element of size 2a x 2b with 9 fdof and 12 ddof is used for discretizing the problem in to desired number of elements. The elemental equilibrium matrix $[B^e]$ is obtained by substituting the value of a and b and are assembled to obtain the global equilibrium matrix.

Step 2: The compatibility matrix is obtained from the displacement deformation relations (DDR) i.e. $\beta = [B]^T\{X\}$. In the DDR, n deformations which correspond to n force variables are expressed in terms of m displacements. The problem requires r = n – m compatibility conditions [C] that are obtained by eliminating the m displacements from the n DDR's. These are obtained by using auto-generated Matlab based computer program by giving input as upper part of the global equilibrium matrix.

Step 3: The flexibility matrix for the problem is obtained by diagonal concatenation of the elemental flexibility matrices.

Step 4: By multiplying compatibility matrix [C] and global flexibility matrix [G] bottom most part of the global equilibrium matrix is obtained. Assembling both gives complete [S] matrix of size n x n, which comprises of equilibrium equations and compatibility conditions. The forces are then obtained by using Matlab's inverting procedure.

Step 5: The displacements are calculated by using relation ($\{X\} = [J][G]\{F\}$), where [J] = m rows of matrix $[[S]^{-1}]^{T.}$

IV. NUMERICAL EXAMPLES

In order to test the accuracy and usefulness of the integrated force method, it is applied to a variety of problems. All the plates are considered of steel with modulus of elasticity as 2.01×10^{11} N/m^2 and Poisson's ratio as 0.3. For all the problems, solution is obtained using a software developed in VB.Net using appropriate data as per loading and boundary conditions. For the development of compatibility conditions and for the solution of equations which involves both equilibrium and compatibility conditions Matlab software is used. Matlab software is also used for plotting the moment contours and deflection profiles. The following applications are intended to demonstrate the capability of the suggested formulation.

1) Square Plate under Patch Loading

A simply supported square plate of size 4000 mm x 4000 mm x 200 mm is studied under a central patch loading of intensity 10 kN/m^2 over an area 1600 mm x 1600 mm as shown in Fig. 1. Due to two way symmetry only quarter of the plate is analysed by discretizing into 5 x 5 mesh. Results are obtained for Mx, My and Mxy and w, Θ_x and Θ_y at all the nodes. Some of the results are presented here in Table I and are compared with the classical results [2]. Also contours of Mx and deflection profile are included here in Figs. 2 and 3 respectively.

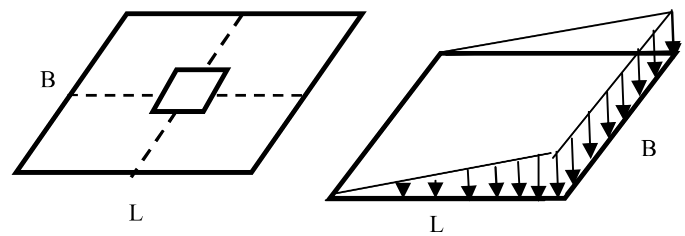

Fig. 1 Plate under patch and hydrostatic loading.

TABLE I
RESULTS FOR SQUARE PLATE UNDER PATCH LOADING

Node	Deflection w in mm		Moment Mx in N-m	
	IFM	EXACT	IFM	EXACT
1	0.0000	0.0000	0.00	0.00
2	0.0073	0.0079	956	967
3	0.0148	0.0151	1838	1840
4	0.0198	0.0207	2501	2533
5	0.0237	0.0241	2956	2987
6	0.0249	0.0252	3102	3132

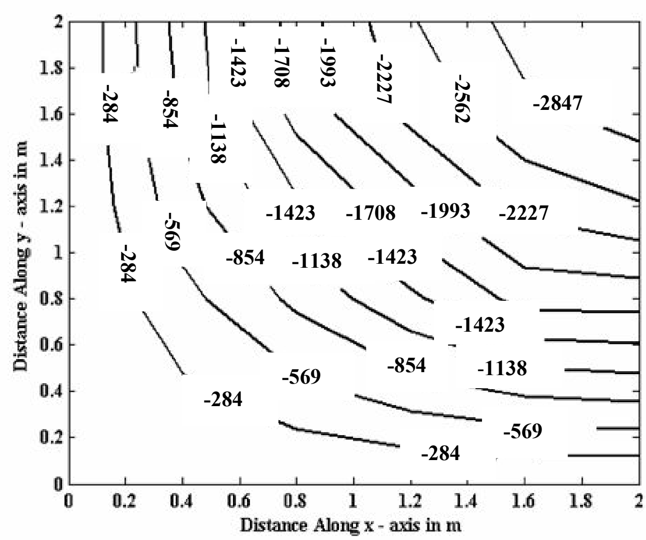

Fig. 2 Contours of Mx for plate under patch loading.

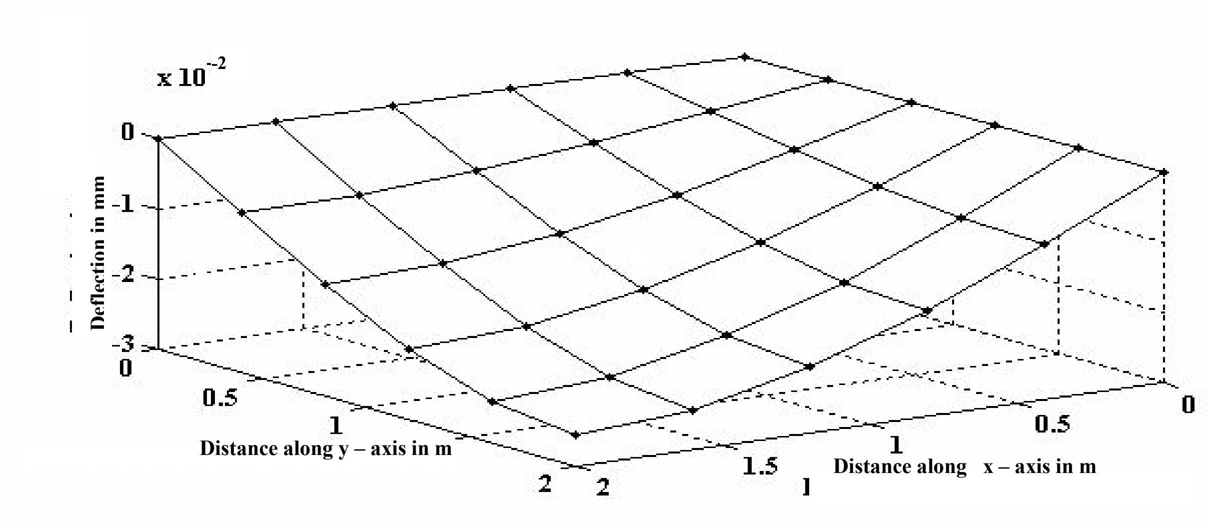

Fig. 3 Deflection profile of quarter plate under patch loading.

2) Rectangular Plate under Patch Loading

Next, a rectangular plate of size 6000 mm x 4000 mm x 200 mm simply supported along all edges is considered under partial patch loading of intensity of 10kN/m^2 over a size 2400 mm x 1600 mm. Results obtained for w displacement and moment Mx at all the nodes lying on the central line along x direction are presented here in Table II and are also plotted in Fig. 4 for the quarter rectangular plate along with the results for square plate.

TABLE II
RESULTS FOR RECTANGULAR PLATE UNDER PATCH LOADING

Node	Deflection w in mm		Moment Mx in N-m	
	IFM	EXACT	IFM	EXACT
1	0.0000	0.0000	0.00	0.00
2	0.0133	0.0153	998	1055
3	0.0237	0.0286	2067	2098
4	0.0349	0.0393	2689	2706
5	0.0422	0.0463	3210	3246
6	0.0448	0.0487	3392	3412

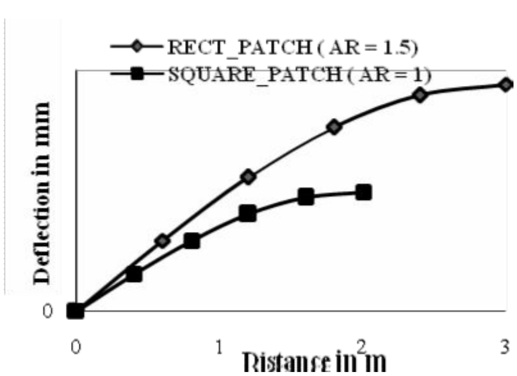

Fig. 4 Variation of deflection w along central line.

3) Square Plate under Hydrostatic Loading

The third example considered here is that of a simply supported square plate of size 4000 x 4000 x 200 mm subjected to a hydrostatic loading (uniformly varying lateral load) of intensity 10 kN/m^2. Due to one way symmetry, only half of the plate is analysed by discretizing it into 10 x 5 grid. Results obtained for lateral displacement w and moment Mx at the nodes lying in the x – direction on the central line are reported here in Table III and are compared with the available classical solution [2]. Moment Mx contours are depicted in Fig. 5 whereas deflection profile is shown in Fig. 6.

TABLE III
RESULTS FOR SQUARE PLATE UNDER HYDROSTATIC LOADING

Node	Deflection w in mm		Moment Mx in N-m	
	IFM	EXACT	IFM	EXACT
1	0.0000	0.0000	0.00	0.00
2	0.0094	0.0097	827	852
3	0.0182	0.0188	1635	1687
4	0.0258	0.0265	2452	2492
5	0.0314	0.0323	3184	3218
6	0.0344	0.0354	3800.	3853
7	0.0343	0.0355	4211	4224
8	0.0308	0.0317	4265	4297
9	0.0235	0.0242	3765	3790
10	0.0128	0.0132	2479	2512
11	0.0000	0.0000	0.0000	0.000

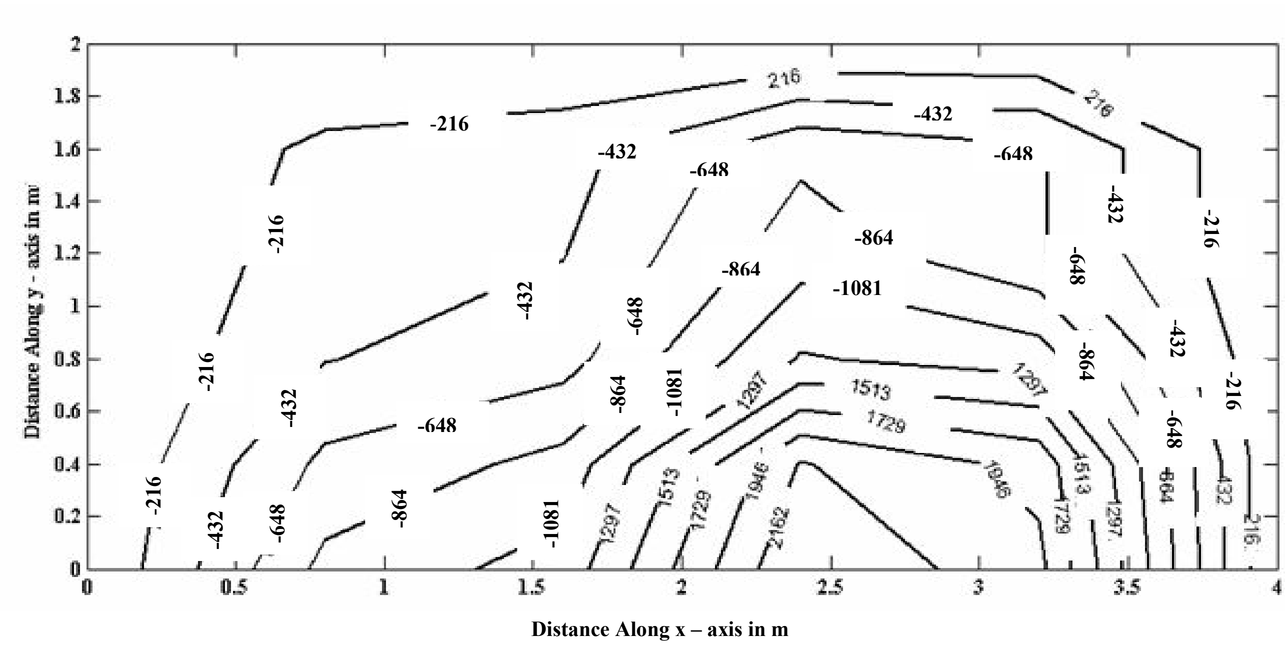

Fig. 5 Contours of Mx for plate under hydrostatic loading.

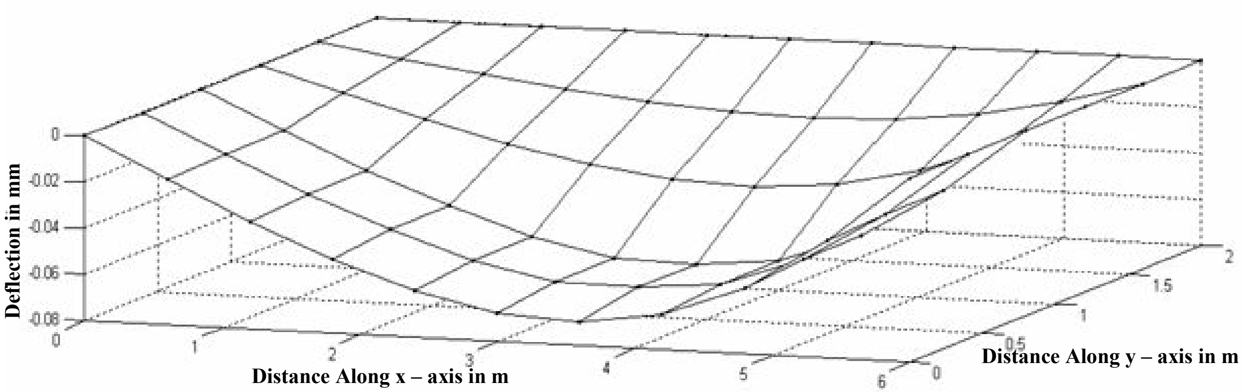

Fig. 6 Deflection profile of half plate under hydrostatic loading.

4) Rectangular Plate under Hydrostatic Loading

Finally, a rectangular plate of size 6000 x 4000 x 200 mm is solved under a linearly varying load of intensity 10kN/m^2 in the x direction. Again, one way symmetry is used to discretized 6000 mm x 2000 mm portion of the plate into 10 x 5 grid. Some of the results obtained using the integrated force method are compared in Table IV with those obtained using the analytical method [3]. The values of lateral deflection w and moment Mx are reported in the table for nodes 1 to 11 lying on the central line of the plate in the x direction. Also, variation of deflection w is depicted in Fig. 7 for square plate and rectangular plate having aspect ratio (AR) as 1.0 and 1.5 respectively

TABLE IV
RESULTS FOR RECTANGULAR PLATE UNDER HYDROSTATIC LOADING

Node	Deflection w in mm		Moment Mx in N-m	
	IFM	EXACT	IFM	EXACT
1	0.0000	0.0000	0.00	0.00
2	0.0163	0.0170	753	778
3	0.0319	0.0333	1529	1560
4	0.0459	0.0479	2300	2374
5	0.0572	0.0597	3054	3177
6	0.0646	0.0674	3899	3998
7	0.0668	0.0697	4582	4683
8	0.0621	0.0648	5002	5169
9	0.0492	0.0514	4872	4973
10	0.0278	0.0290	3549	3667
11	0.0000	0.0000	0.0000	0.000

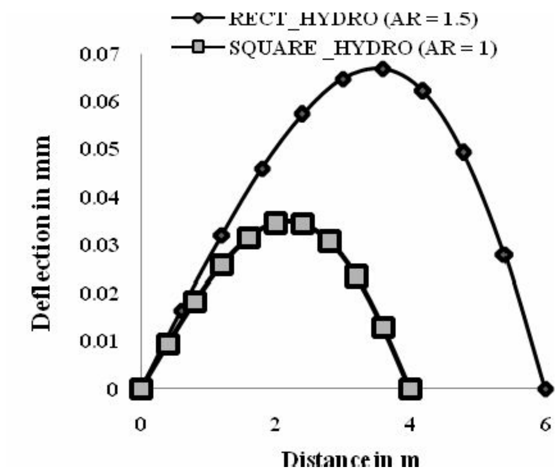

Fig. 7 Variation of displacement w along central line.

V. CONCLUSIONS

1. Analysis of rectangular plates subjected to patch and hydrostatic loading is presented by an integrated force approach which combines equilibrium and compatibility conditions in a single set of equations. A four nodded rectangular element with 9 force and 12 displacement degrees of freedom is used with 5 x 5 discretization to solve patch loading examples whereas 10 x 5 discretization is used to get the solution of square and rectangular plates under hydrostatic loading. In all cases, results are found in good agreement with those based on classical formulation.

2. Interfacing a program developed in VB.Net with Matlab software has not only helped in generating the compatibility conditions and solution of equations but also in producing attractive plots such as moment contours and deflection profiles for both square and rectangular plates when subjected to lateral loading.

3. In the present paper, to take advantage of symmetry and thus to reduce the total number of unknowns, plates subjected to only central patch loading were analysed. However, the integrated force method, with appropriate discretization, can be used to solve any rectangular plate problem subjected to any size patch load acting anywhere on the plate.

4. In case of hydrostatic loading only simply supported edge condition was considered. The suggested method, however, is capable to simulate any type of boundary condition required in modeling the plates used in different types of liquid containers.

VI. REFERENCES

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